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A Mixed Integer Linear Program for Synchronizing Traffic Signals for Maximal Bandwidth

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Abstract

Traffic signals can be synchronized so that a car, starting at one end of the street and traveling at preassigned speeds, can go to the other end without stopping for a red light. The portion of a signal cycle for which this is possible is called the bandwidth for that direction. Ordinarily the bandwidth in each direction is single, i.e., is not split into two or more intervals within a cycle. For this case we formulate the following problem as a mixed integer linear program: Given an arbitrary number of signals, red times for each signal, upper and lower limits on signal period, upper and lower limits on speed each way between adjacent signals, limits on change in speed and a fixed constant of proportionality between the two bandwidths, find a common signal period, speeds between signals, and the relative phasing of the signals so as to maximize the sum of the bandwidths.

1. Introduction

In an earlier paper [1] J. Morgan and the author have developed an algorithm for solving the following two problems in synchronizing traffic signals along an arterial street:

1. Given an arbitrary number of signals, a common signal period, the green and red times for each signal, and specified travel times between adjacent signals, synchronize the signals to produce bandwidths that are equal in each direction and as large as possible.

2. Adjust the synchronization to increase one bandwidth to some specified, feasible value and maintain the other as large as is then possible.

Subsequently, R. Oliver pointed out to the author that a two-signal version of problem 1 could be set up as a linear program. We here expand this idea into a rather general formulation of the maximal bandwidth problem. The formulation comes out as a fairly sizable mixed integer linear program. For problems 1 and 2 this offers no advantages and many disadvantages. However, the linear programming format opens up the possibility of solving more general problems involving the introduction of new decision variables.

For example, maximal bandwidth calculations as usually performed have a disconcerting feature. On a long street the critical signals that construct bandwidth may turn out to be very far apart. Then a small change in one of the speeds along the street is likely, upon re-solving the problem, to result in a different synchronization and a different bandwidth. Yet, drivers do not hold their speeds particularly constant and, indeed, as is well known, tend to adjust their speeds to the signals. Thus, it makes a good deal of sense to let speed between signals be a variable. At the same

time, it is a fairly simple matter in a linear program to put limits on the speeds that can be assumed.

Another variable that can be introduced explicitly in a linear program is the signal period. In Problem 1, period is a constant and, although it is not too difficult to use our earlier methods to examine a considerable number of values in an organized way, the continuous variation of the linear program formulation seems preferable.

2. Definitions

Consider a two-way street having n traffic signals. Directions on the street will be identified as outbound and inbound. The signals will be denoted $S_1, S_2, \ldots, S_n$ with the subscript increasing in the outbound direction.

Under some circumstances, a car can start at one end of the street and, by traveling at preassigned speeds, go to the other end without stopping. The portion of a signal cycle during which this is possible will be called the bandwidth in this direction. (Ordinarily the bandwidth in each cycle is single, i.e., is not split into two or more intervals within a cycle. We restrict ourselves to this case.)

Figure 1 shows a space-time diagram for travel on a street. Heavy horizontal lines indicate when the signals are red. The zig-zag lines represent trajectories of cars passing unimpeded along the street in the directions shown. Changes in slope correspond to changes in speed. The set of possible unimpeded trajectories in a given direction forms a green band whose horizontal width is the bandwidth for that direction. Although drawn but once, it is clear that the green bands appear once per cycle in parallel bands across the diagram.

Certain signals, usually two or three of the n, form the ultimate limitation or bandwidth and will be called critical signals. A signal S_j is said to be a critical signal if one side of S_j's red touches the green

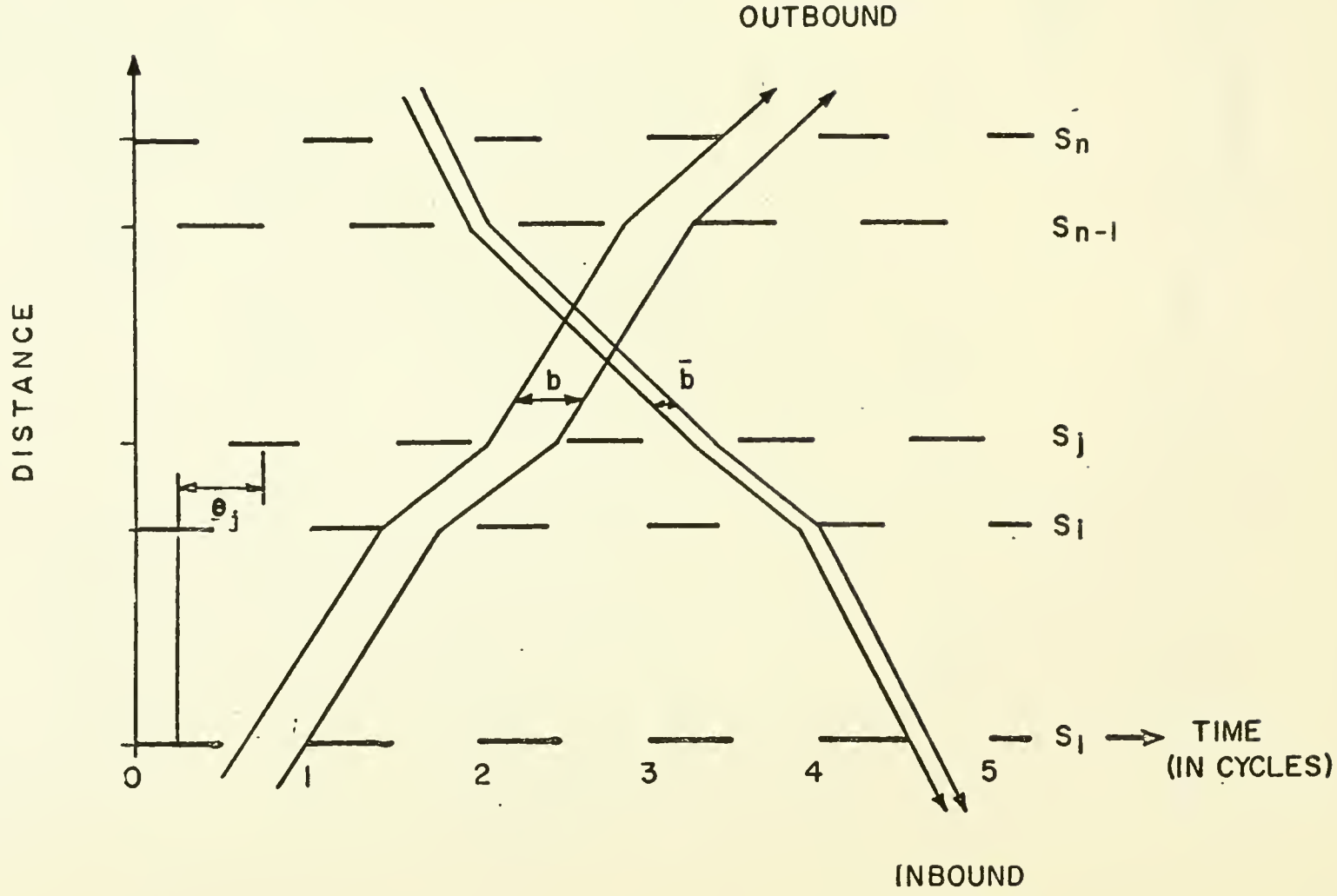

Figure 1. Space-time diagram

band in one direction and the other side touches the green band in the other direction. Thus, in Figure 1 signals S_1 and S_i are cirtical, but no others are.

3. Basic Maximal Bandwidth LP

First we set up the basic Problem 1 above as a mixed integer linear program.

3.1 Notation

Let

r_i = red time of S_i on street under study. (cycles)

b ($\bar{b}$) = outbound (inbound) bandwidth. (cycles)

t_i ($\bar{t}_i$) = travel time from S_1 to S_i in outbound direction (travel time from S_i to S_1 in inbound direction). (cycles)

ϕ_i ($\bar{\phi}_i$) = distance from center of red at S_1 to the center of a particular red at S_i. See Figure 2. The two reds are chosen so that each is immediately to the left (right) of the same outbound (inbound) green band. ϕ_i ($\bar{\phi}_i$) is positive if S_i's center of red lies to the right (left) of S_1's. (cycles)

w_i ($\bar{w}_i$) = distance from the right (left) side of S_i's red to the green band. See Figure 2. (cycles)

$m_j = \phi_j + \bar{\phi}_j$

Note that a quantity having dimensions of time can always be expressed in cycles by dividing by the period.

3.2 Formulation

From Figure 2 can be read the identities

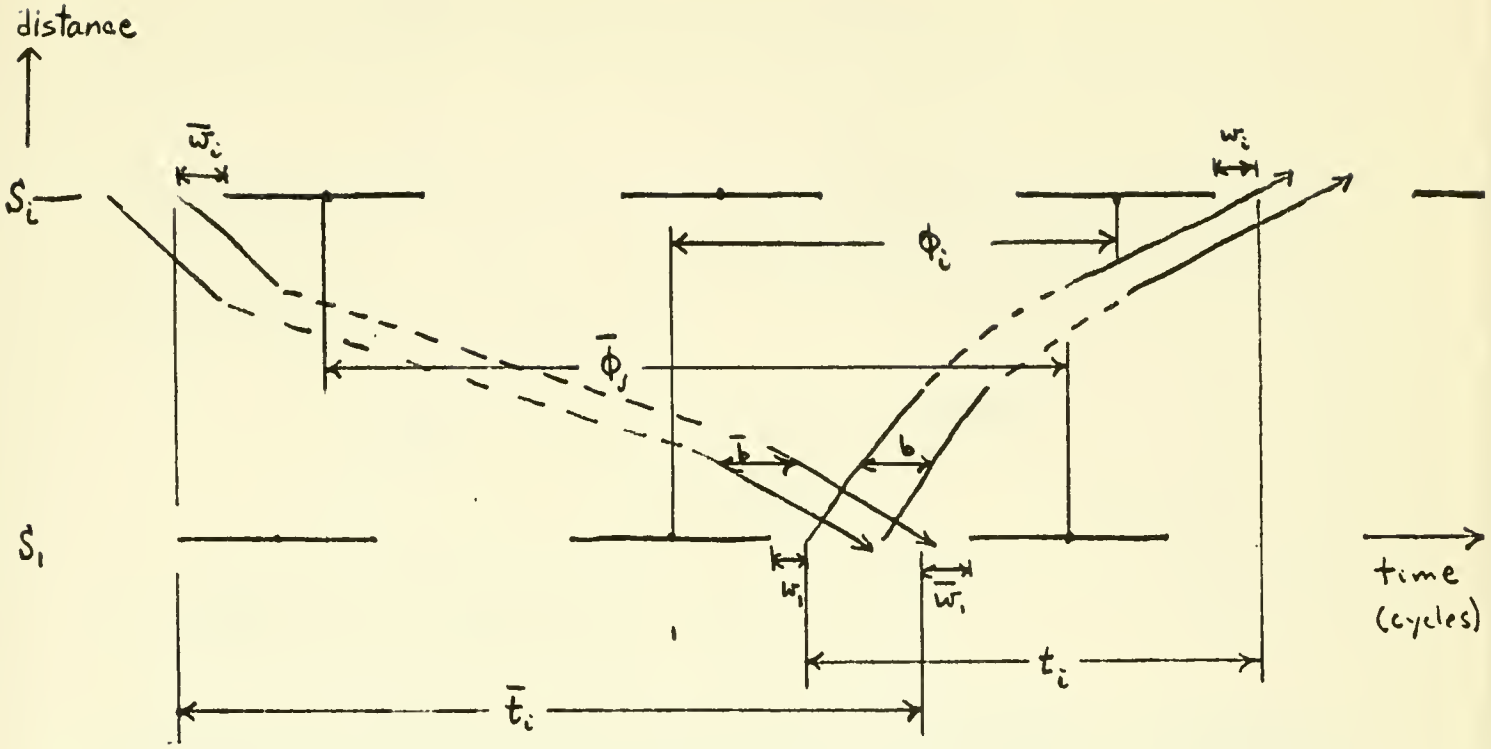

Figure 2. Geometry of the green bands. Notice that $\phi_j + \bar{\phi}_j$ must be an integral number of cycles

$$(1/2)\ r_1 + w_1 + t_i - w_i - (1/2)r_i = \phi_i$$

$$(1/2)\ r_1 + \bar{w}_1 + \bar{t}_i - \bar{w}_i - (1/2)r_i = \bar{\phi}_i\ .$$

The values of ϕ_i and $\bar{\phi}_i$ have the very important constraint that their sum is an integer, but otherwise they are unrestricted. Therefore, the above two constraints can be replaced by

$$(w_1+\bar{w}_1) - (w_i+\bar{w}_i) - m_i = (r_1-r_1) - (t_i+\bar{t}_i)$$

$$m_i = \text{integer}\ .$$

From Figure 2 we also see that

$$w_i + b \leq 1 - r_i$$

and that a similar expression holds for the inbound direction. Consequently, Problem 1 can be represented by the linear program:

LP1. Find b, $\bar{b}$, w_i, $\bar{w}_i$, m_i to

$$\text{Max } (b + \bar{b})$$

Subject to:

$$b = \bar{b} \qquad \text{(LP1.1)}$$

$$w_i + b \leq 1 - r_i \qquad \text{(LP1.2a)}$$

$$\bar{w}_i + \bar{b} \leq 1 - r_i \qquad \text{(LP1.2b)}$$

$$\left.\begin{array}{l} \text{(2a) and (2b)} \\ \text{(3)} \end{array}\right\} \quad i = 2,\ldots,n$$

$$(w_1+\bar{w}_1) - (w_i+\bar{w}_i) - m_i = (r_i-r_1) - (t_i+\bar{t}_i) \qquad \text{(LP1.3)}$$

$$m_i = \text{integer} \qquad \text{(LP1.4)}$$

$$b,\ \bar{b},\ w_i,\ \bar{w}_i \geq 0$$

LP1 has 3n equations and 3n+1 unknowns, not counting slack or artificial variables.

4. More Decision Variables

Next we let period and speed be variables. Each will be constrained by upper and lower limits. In addition changes in speed from one street

segment to the next will be limited. Finally, instead of requiring equal bandwidths in each direction, we generalize to let one bandwidth be a fixed proportion of the other.

4.1 Notation

Let

k = constant of proportionality between b and $\bar{b}$.

T = signal period. (seconds)

d_i = distance from S_{i-1} to S_i, $i = 2,\ldots, n$. (meters)

v_i ($\bar{v}_i$) = speed between S_{i-1} and S_i outbound (S_i and S_{i-1} inbound),

$i = 2,\ldots, n$. (meters/second)

We wish to put limits on speed, say,:

$$e_i \leq v_i \leq f_i \qquad i = 2,\ldots,n ,$$

or, equivalently,

$$\frac{1}{f_i} \leq \frac{1}{v_i} \leq \frac{1}{e_i} .$$

We also wish to constrain change of speed. This will be done by constraining change in reciprocal speed. Although the two are not quite the same, constraining change in the reciprocal surely satisfies the basic intention, which is to have a means of preventing large abrupt speed changes. For notation, we use

$$\frac{1}{h_i} \leq \frac{1}{v_i} - \frac{1}{v_{i-1}} \leq \frac{1}{g_i} \qquad i = 3,\ldots, n .$$

Limits for T can be denoted

$$T_1 \leq T \leq T_2$$

Next, let

y_i ($\bar{y}_i$) = time to travel from S_{i-1} to S_i outbound (S_i to S_{i-1} inbound),

$i = 2,\ldots,n.$ (cycles)

$z = 1/T$ = signal frequency (cycles/second)

Then

$$y_i = \frac{d_i}{v_i T}$$

$$(d_i/f_i)z \leq y_i \leq (d_i/e_i)z$$

$$(d_i/h_i)z \leq y_i - y_{i-1} \leq (d_i/g_i)z$$

$$t_i = \sum_{k=2}^{i} y_k \qquad i = 2, \ldots, n$$

Similar expressions can be written involving $\bar{y}_i$ and $\bar{t}_i$. Finally, the constraint relating b and $\bar{b}$ is

$$\bar{b} = k\, b$$

4.2 Formulation

The above can be assembled into

LP2. Find b, $\bar{b}$, w_i, $\bar{w}$, y_i, $\bar{y}_i$, m_i to

$$\text{Max } (b + \bar{b})$$

Subject to:

$$\bar{b} = k\, b \qquad \text{(LP2.1)}$$

$$(1/T_2) \leq z \leq (1/T_1) \qquad \text{(LP2.2)}$$

$$w_i + b \leq 1 - r_i \qquad \text{(LP2.3a)}$$

$$\bar{w}_i + \bar{b} \leq 1 - r_i \qquad \text{(LP2.3b)}$$

$i = 1, \ldots, n$

$$(w_1 + \bar{w}_1) - (w_i + \bar{w}_i) - m_i + \sum_{k=2}^{i} (y_k + \bar{y}_k) = (r_1 - r_1) \qquad \text{(LP2.4a)}$$

$$m_i = \text{integer} \qquad \text{(LP2.4b)}$$

$$(d_i/f_i)\, z \leq y_i \leq (d_i/e_i)\, z \qquad \text{(LP2.5a)}$$

$$(d_i/\bar{f}_i)\, z \leq \bar{y}_i \leq (d_i/\bar{e}_i)\, z \qquad \text{(LP2.5b)}$$

$i = 2, \ldots, n$

$$(d_i/h_i)\ z \le y_i - y_{i-1} \le (d_i/g_i)\ z \qquad \text{(LP2.6a)}$$

$$(d_i/\bar{h}_i)\ z \le \bar{y}_i - \bar{y}_{i-1} \le (d_i/\bar{g}_i)\ z \qquad \text{(LP2.6b)}$$

$i=3,\ldots n$

$$b,\ \bar{b},\ w_i,\ \bar{w}_i,\ y_i,\ \bar{y}_i,\ \ge 0$$

LP2 involves (11n-10) constraints and 4n variables, not counting slack and artificial variables

5. The Synchronization

The linear programming variables determine the synchronization of the signals. Let

θ_j = relative phase of S_1 and S_j, measured as the time from the center of a red of S_1 to the next center of red of S_j. (cycles) By convention $0 \le \theta_j < 1$. See figure 1.

A set of θ_j, $j = 2,\ldots, n$, will be called a synchronization.

From Figure 2, we see

$$\theta_i = \text{man } \phi_i$$

where "man" stands for "mantissa of" (e.g., man (5.2) = .2, man (-1.3) = .7).

Here

$$\phi_i = 1/2(r_1 - r_i) + w_1 + t_i - w_i .$$

This can easily be computed from the linear program results.

References

1. J. T. Morgan and J. D. C. Little, "Synchronizing Traffic Signals for Maximal Bandwidth," Operations Research, 12, 896-912, (Nov-Dec. 1964).

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